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Finite-size corrections for nested Bethe ansatz models and conformal invariance

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Abstract. The method presented here permits us to systematically compute physical magnitudes for large but finite size in theories solvable by the nested Bethe ansatz. The value of the central charge c and the surface tension are explicitly calculable in this way. For the fundamental vertex and spin models associated to simply laced Lie algebras, c turns out to be equal to the rank of the algebra.

1. Introduction

Integrable theories are those possessing as many commuting and conserved physical quantities as degrees of freedom, so there is an infinite number in the thermodynamical limit. The explicit solution of these models is to a large extent possible through the use of the Bethe ansatz and its generalisations [1].

The Bethe ansatz for the eigenvectors lead a set of algebraic equations whose roots determine the common eigenvalues of the commuting operators of the theory including the Hamiltonian. In the thermodynamic limit the number of equations and roots tends to infinity and the Bethe ansatz equation (BAE) usually yields a set of linear integral equations for the densities of roots. Fourier transforms solve the equation in this limit. However, the explicit resolution of the BAE for a finite-size N is a formidable problem. Besides its interest from the point of view of integrable theories, the finite-size resolution of these models provides deep insights on their conformal content when they are gapless. For models with non-zero gap it gives information on the surface tension.

A method to systematically compute large but finite-size corrections in integrable theories was presented in [2] for the six- and eight-vertex [3] models and the associated magnetic chains. These methods are elaborated and developed in [4-7].

The aim of this paper is to generalise the method of [2] to theories solvable by the nested Bethe ansatz (NBA), i.e. theories with the following internal structure: vertex models where the links can be in q different states (with $q > 2$) or q-component spin models. The field theoretical models solvable by NBA usually exhibit non-Abelian internal symmetries like the chiral fermionic models of [8-10], the Gross-Neveu model and the sigma models connected with multiflavour fermionic models [11].

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The calculation method presented here is based on a new integral representation for the physical quantities (such as the energy) which are exact for all N (the number of sites). These integral representations generalise those of [2] and [3] and they contain functions $\sigma_N^{(l)}(\lambda)$ ($1 \leq l \leq r$, r = number of steps in the NBA) that can be thought of as finite-N generalisations of the densities of roots. We find that the contribution of the different steps of the NBA simply adds. The expressions obtained appear as infinite series (equations (2.31)-(2.33)) that can be summed up as geometric series. Then the large-N behaviour follows by the saddle-point method. In massive models (non-zero gap) the relevant saddle points are complex roots of

$$\sigma_\infty^{(l)}(\lambda) = 0 \qquad 1 \leq l \leq r. \tag{1.1}$$

They dominate the large-N behaviour of all physical quantities. For gapless theories, endpoints of integration ($\lambda = \pm\infty$) govern the large-N regime. In the first case, the finite-N corrections are exponentially small in N and in the second case they are of power type. In both cases, closed-form expressions can be explicitly derived. In the transition between the two regimes, logarithmic terms also appear [6] (already with a one-step Bethe ansatz).

The method presented in § 2 is valid for any theory solvable by the Bethe ansatz where the ground state is formed by real roots.

We apply this method to a large class of fundamental vertex models associated to simple Lie algebras. The NBAE for all these models can be written systematically in Lie algebraic terms [12] (many-body integrable systems related to Lie algebras can be found in [13]). For the models associated to simply laced Lie algebras (A_n, D_n, E_6, E_7, E_8) (see table 1) we find a simple expression for the dominant finite-size corrections to the free energy:

$$f_N - f_\infty = -\pi r/6N^2 + o(1/N^2) \tag{1.2}$$

for periodic boundary conditions. Here r stands for the rank of the algebra.

The large-size behaviour of the free energy for a conformally invariant model is predicted by modular invariance to be [14]

$$f_N - f_\infty = -\pi c/6N^2 + o(1/N^2) \tag{1.3}$$

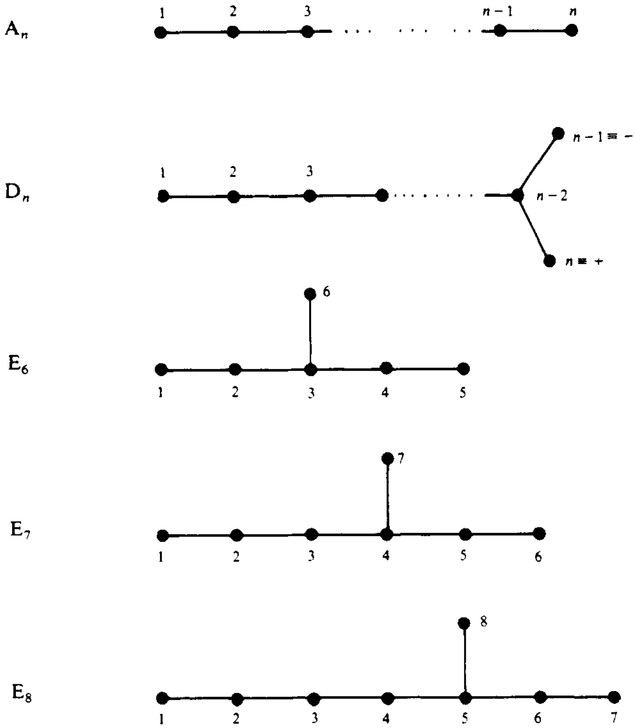
where c is the central charge in the Virasoro algebra and PBC are used. Comparison between (1.2) and (1.3) shows that $c = r$ for the fundamental vertex model associated to simply laced Lie algebras. Equation (1.2) also holds for the $U(1)^r$ symmetric model of [1e] and [15] in its gapless regime. Hence c is still equal to r although this model is not $U(r+1)$ invariant but only $U(1)^r$ invariant.

When the mass gap is non-zero in the thermodynamic limit conformal invariance is clearly absent. One finds in this case asymptotically degenerate eigenvalues for the transfer matrix [1c, 16]. Their ratio can be related through standard arguments to the interfacial tension [1c]. We compute the interfacial tension (S) in this way for the model of [1e] and [15] in § 4. It turns out that S follows from the zeros of an elliptic function (equation (4.19)). Explicit expressions of S for large and small anisotropies (γ) are derived (equations (4.22), (4.23) and (4.26)). The results presented generalise for this $q(2q-1)$-vertex model the formulae of Baxter for the six-vertex model ($q = 2$) [1c].

Scaling hypotheses predict that S should be related to the correlation length ξ by [17]

$$S\xi = 1 \tag{1.4}$$

Table 1. Simply laced Lie algebras



in our units. (This relation is explicitly verified in six- and eight-vertex models [1c, 18]). Hence, as a by-product of our S calculation in § 4 one obtains in addition using equation (1.4) the correlation length ξ for the $q(2q-1)$-vertex model of [1e] and [15].

2. Finite-size solution of nested Bethe ansatz models

A method of computing finite-size corrections for integrable theories solvable by the nested Bethe ansatz (NBA) is presented in this section. Specific applications are presented in § 4.

Generally speaking, the free energy for such models for a large but finite number of sites N is

$$f_N(\theta) = -(1/N)\log \Lambda_{\max}(\theta) + O(e^{-\delta N}). \tag{2.1}$$

Here $\Lambda_{\max}$ is the maximum eigenvalue of the transfer matrix $\tau(\theta)$, θ is the spectral parameter and $\delta > 0$ is related to the next-to-leading eigenvalue of $\tau(\theta)$. Then

$$\log \Lambda_{\max} = \mathrm{i} \sum_{k=1}^{r} \sum_{j_k=1}^{p_k} \phi(\lambda_{j_k}^{(k)} + \mathrm{i}\theta, \omega_k) \tag{2.2}$$

where the functions $\phi(z, \alpha)$ are given in § 3, k labels the steps of the NBA ($1 \leq k \leq r$), and the $\lambda_{j_k}^{(k)} j_k = 1, \ldots, p_k$ are the roots of the NBA equations

$$\sum_{k=1}^{r} \sum_{j_k=1}^{p_k} \phi(\lambda_{j_l}^{(l)} - \lambda_{j_k}^{(k)}, \omega_{lk}) = N\phi(\lambda_{j_l}^{(l)}, \omega_l) - 2\pi I_{j_l}^{(l)}. \tag{2.3}$$

The parameters $\omega_{lk} = (\alpha_l, \alpha_k)$ and $\omega_k = (\omega, \alpha_k)$ are given and they are related to roots and weights of the underlying Lie algebra (see § 3). The $I_{j_k}^{(l)}$ are half-odd integers.

It is convenient to introduce the functions

$$t_N^{(l)}(\lambda) = \phi(\lambda, \omega_l) - \frac{1}{N} \sum_{k=1}^{r} \sum_{j_k=1}^{p_k} \phi(\lambda - \lambda_{j_k}^{(k)}, \omega_{l_k}). \tag{2.4}$$

They are continuous functions of λ for real λ. At the roots of the NBAE (2.3)

$$t_N^{(l)}(\lambda_{j_l}^{(l)}) = (2\pi/N) I_{j_l}^{(l)}. \tag{2.5}$$

Now define

$$\sigma_N^{(k)}(\lambda) = \frac{1}{2\pi} \frac{\mathrm{d}t_N^{(k)}}{\mathrm{d}\lambda}. \tag{2.6}$$

The half-odd integers $I_{j_k}^{(k)}$ form for fixed k a monotonic sequence for the antiferromagnetic vacuum

$$I_{j_k+1}^{(k)} - I_{j_k}^{(k)} = +1 \qquad 1 \leq k \leq r, 1 \leq j_k \leq p_k. \tag{2.7}$$

For excited states these sequences exhibit jumps for some values of j_k:

$$I_{j_k+1}^{(k)} - I_{j_k}^{(k)} = 1 + \sum_{h=1}^{N_h^{(k)}} \delta_{jk, t_h^{(k)}}. \tag{2.8}$$

The values of $\lambda_{j_k}^{(k)}$ associated to these missing half integers are called holes and are denoted by $\theta_h^{(k)}$:

$$t_N^{(k)}(\theta_h^{(k)}) = (2\pi/N)(1 + I_{t_h^{(k)}}^{(k)}). \tag{2.9}$$

When N goes to infinity, the p_k also tend to infinity for antiferromagnetic states and the $\lambda_{j_k}^{(k)}$ tend to have a continuous distribution for each value of k with density

$$\rho^{(k)}(\lambda_{j_k}^{(k)}) = \lim_{N \to \infty} \frac{1}{N(\lambda_{j_k+1}^{(k)} - \lambda_{j_k}^{(k)})}. \tag{2.10}$$

One finds from (2.5) and (2.6), for large N,

$$\sigma_\infty^{(k)}(\lambda_{j_k}^{(k)}) = \lim_{N \to \infty} \frac{I_{j_k+1}^{(k)} - I_{j_k}^{(k)}}{N(\lambda_{j_k+1}^{(k)} - \lambda_{j_k}^{(k)})}. \tag{2.11}$$

Then (2.8)-(2.11) yield for large N

$$\sigma_\infty^{(k)}(\lambda) = \rho_{(\lambda)}^{(k)} + \frac{1}{N} \sum_{h=1}^{N_h^{(k)}} \delta(\lambda - \theta_h^{(k)}). \tag{2.12}$$

The NBA, equation (2.3), admit both real and complex roots. We discuss in the present paper effects related to real roots. Taking the difference in that case between (2.3) for $j_n = l_{n+1}$ and $j_n = l_n$ yields in the $N = \infty$ limit

$$\rho^{(j)}(\lambda) - \sum_{l=1}^{r} \int_{-a_l}^{a_l} \mathrm{d}\mu \, K_{jl}(\lambda - \mu) \rho_{(\mu)}^{(l)} = \frac{\phi'(\lambda, \omega_j)}{2\pi} - \frac{1}{N} \sum_{h=1}^{N_h^{(j)}} \delta(\lambda - \theta_h^{(j)}) \tag{2.13}$$

or

$$\sigma_\infty^{(j)}(\lambda)-\sum_{l=1}^{r}\int_{-a_l}^{a_l}\mathrm{d}\mu\, K_{jl}(\lambda-\mu)\rho^{(l)}(\mu)=\frac{\phi'(\lambda,\omega_j)}{2\pi}-\frac{1}{N}\sum_{l=1}^{r}\sum_{h=1}^{N_h^{(l)}}K_{jl}(\lambda-\theta_h^{(l)}) \tag{2.14a}$$

where

$$K_{jl}(\lambda)=2\pi\phi'(\lambda,\omega_{jl}). \tag{2.14b}$$

The following relation was used to derive (2.13) and (2.14a):

$$\lim_{N\to\infty}\frac{1}{N}\sum_{j_k=1}^{p_k}f(\lambda_{j_k}^{(k)})=\int_{-a_k}^{a_k}\mathrm{d}\lambda\, f(\lambda)\rho_{(\lambda)}^{(k)}. \tag{2.15}$$

Here a_j may be finite or infinite. Equations (2.13) and (2.14a) can be solved by the Fourier expansion

$$\sigma_\infty^{(l)}(\mu)=\int_{-\infty}^{+\infty}\frac{\mathrm{d}x}{2\pi}\exp(\mathrm{i}\mu x)\hat{\sigma}^{(l)}(x) \tag{2.16}$$

when $a_l=\infty$ or

$$\sigma_\infty^{(l)}(\mu)=\sum_{n\in z}\exp(2\mathrm{i}m\mu)\hat{\sigma}_n^{(l)} \qquad \text{for } a_l=\pi/2.$$

It is convenient to express the solutions of (2.14a) and (2.14b) in terms of the resolvent $R_{lk}(\lambda)$, i.e. the solution of the equation

$$R_{lk}(\lambda)-\sum_{j=1}^{r}\int_{-a_k}^{a_k}\mathrm{d}\mu K_{lj}(\lambda-\mu)R_{jk}(\mu)=\delta_{lk}\delta(\lambda) \qquad 1\leqslant k,\, l\leqslant r. \tag{2.17}$$

One finds in Fourier space

$$\hat{R}_{lk}(x)=[(1-\hat{K}(x))^{-1}]_{lk} \qquad 1\leqslant k,\, l\leqslant r. \tag{2.18}$$

Explicit expressions for this inverse matrix are available for each specific model (§ 4). Then the densities $\sigma^{(l)}(\lambda)$ are

$$\sigma_\infty^{(l)}(\lambda)=\sigma_v^{(l)}(\lambda)+\frac{1}{N}\sum_{j=1}^{r}\sum_{h}^{N_h^{(j)}}\sigma_{lj}(\lambda-\theta_h^{(j)}) \tag{2.19}$$

where

$$\sigma_v^{(l)}(\lambda)=\sum_{k=1}^{r}\int_{-a_k}^{a_k}\frac{\mathrm{d}\mu}{2\pi}\phi'(\mu,\omega_k)R_{lk}(\lambda-\mu) \tag{2.20}$$

and

$$\sigma_{lj}(\lambda)=\delta_{lj}\delta(\lambda)-R_{lj}(\lambda). \tag{2.21}$$

Here $\sigma_v^{(l)}(\lambda)$ describes the root density for the antiferromagnetic vacuum at $N=\infty$, i.e. in the absence of holes. In the same limit the free energy per site is from (2.1), (2.2) and (2.15),

$$f_\infty(\theta)=-\mathrm{i}\sum_{k=1}^{r}\int_{-a_k}^{a_k}\mathrm{d}\lambda\, \sigma_\infty^{(k)}(\lambda)\phi(\lambda+\mathrm{i}\theta,\omega_k). \tag{2.22}$$

Now the second term in (2.19) describes the hole contribution to the density of roots. The contribution to the free energy from a single hole at $\phi = \theta_h^{(l)}$ is

$$f_l(\varphi) = \mathrm{i} \sum_{k=1}^{r} \int_{-a_k}^{a_k} R_{lk}(\lambda - \varphi)\phi(\lambda + \mathrm{i}\theta, \omega_k)\, \mathrm{d}\lambda = \mathrm{i}[t_\infty^{(l)}(\varphi + \mathrm{i}\theta) + K_l]. \tag{2.23}$$

The last equality follows from (2.6) and (2.20). K_l is an integration constant (see § 4).

Let us now consider the finite-size corrections to $f_\infty(\theta)$, i.e.

$$L_N(\theta) \equiv f_N(\theta) - f_\infty(\theta). \tag{2.24}$$

One finds, with the help of (2.21), (2.2), (2.15) and (2.22),

$$L_N(\theta) = -\mathrm{i} \sum_{l=1}^{r} \int_{-a_l}^{a_l} \mathrm{d}\mu\, \phi(\mu + \mathrm{i}\theta, \omega_l)\left(\frac{1}{N}\sum_{j_l=1}^{p_l} \delta(\mu - \lambda_{j_l}^{(l)}) - \sigma_N^{(l)}(\lambda)\right)$$
$$-\mathrm{i} \sum_{k=1}^{r} \int_{-a_k}^{a_k} \mathrm{d}\lambda\, \phi(\lambda + \mathrm{i}\theta, \omega_k)[\sigma_N^{(k)}(\lambda) - \sigma_\infty^{(k)}(\lambda)]. \tag{2.25}$$

We now study the difference $\sigma_N^{(k)}(\lambda) - \sigma_\infty^{(k)}(\lambda)$. It follows, using (2.4), (2.6), (2.14*a*) and the resolvent (2.18), that

$$\sigma_N^{(k)}(\lambda) - \sigma_\infty^{(k)}(\lambda)$$
$$= -\sum_{l=1}^{r} \int_{-a_l}^{a_l} \mathrm{d}\mu\, [\delta_{kl}\delta(\lambda - \mu) - R_{kl}(\lambda - \mu)]\left(\frac{1}{N}\sum_{j=1}^{n_l} \delta(\mu - \lambda_{j_l}^{(l)}) - \sigma_N^{(l)}(\mu)\right). \tag{2.26}$$

Equations (2.25) and (2.26) hold for the antiferromagnetic vacuum. In the presence of holes, one finds instead inside the brackets in (2.25) and (2.26)

$$\frac{1}{N}\sum_{j=1}^{n_l} \delta(\mu - \lambda_{j_l}^{(l)}) + \frac{1}{N}\sum_{k=1}^{N_h^{(l)}} \delta(\mu - \theta_h^{(l)}) - \sigma_N^{(l)}(\mu). \tag{2.27}$$

Now, inserting (2.26) into (2.25) yields

$$L_N(\theta) = -\mathrm{i} \sum_{l=1}^{r} \int_{-a_l}^{a_l} \mathrm{d}\mu [t_\infty^{(l)}(\mu + \mathrm{i}\theta) + K_l]\left(\frac{1}{N}\sum_{j_l=1}^{p_l} \delta(\mu - \lambda_{j_l}^{(l)}) - \sigma_N^{(l)}(\mu)\right) \tag{2.28}$$

where (2.23) was used. This formula (2.28) has the right structure in order to evaluate its large-N behaviour. It must be noticed that it is the sum of r terms, so one can apply essentially the same method as in [2] and [3] to each term separately. It can be noticed that a formula analogous to (2.28) holds for excited states: the large round bracket in the right-hand side must be replaced by (2.27).

Generally speaking, it is possible to derive the large-N behaviour of any expression with the form

$$I_N = \sum_{l=1}^{r} \int_{-a_l}^{a_l} \mathrm{d}\mu_l f_l(\mu_l)\left(\frac{1}{N}\sum_{j_l=1}^{p_l} \delta(\mu_l - \lambda_{j_l}^{(l)}) + \frac{1}{N}\sum_{h=1}^{N_h^{(l)}} \delta(\mu_l - \theta_h^{(l)}) - \sigma_N^{(l)}(\mu_l)\right) \tag{2.29}$$

where the function $f_l(\mu)$ is given. Changing the integration variables in (2.29) to $t_l = t_N^{(l)}(\mu_l)$ as defined by (2.4) yields

$$I_N = \sum_{l=1}^{r} \int_0^{Q_l} \frac{\mathrm{d}t_l}{2\pi} f_l(\mu_l(t_l))\left(\frac{1}{N}\sum_{k=1}^{p_l + N_h^{(l)}} \delta(t_l - t_k^{(l)}) - 1\right) \tag{2.30}$$

where

$$t_k^{(l)} = (2\pi/N)(k+\tfrac{1}{2}) \qquad k = 1, 2, \ldots, p_l + N_h^{(l)}$$

$$Q_l = \int_{-a_l}^{a_l} \mathrm{d}\lambda\, \sigma_N^{(l)}(\lambda).$$

Fourier expanding the $\delta(z)$ gives

$$I_N = \sum_{l=1}^{r} \sum_{\substack{\alpha_l=-\infty \\ \alpha_l \neq 0}}^{+\infty} (-1)^{\alpha_l} T^l_{N\alpha_l} \tag{2.31}$$

where

$$T^l_n = \int_{-a_l}^{a_l} \mathrm{d}\lambda\, f_l(\lambda) \sigma_N^{(l)} \exp(\mathrm{i} n t_N^{(l)}(\lambda)). \tag{2.32}$$

Now, in order to find the dominant large-N behaviour, one can replace

$$T^l_n \to T^{l,as}_n \equiv \int_{-a_l}^{a_l} \mathrm{d}\lambda\, f_l(\lambda) \sigma_\infty^{(l)}(\lambda) \exp(\mathrm{i} n t_\infty^{(l)}(\lambda)). \tag{2.33}$$

The right-hand side of (2.33) is exactly known from (2.18)-(2.21). One must take

$$f_l(\lambda) = -\mathrm{i}[t_\infty^{(l)}(\lambda + \mathrm{i}\theta) + K_l] \tag{2.34}$$

to compute finite-size corrections to the free energy.

For large n, $T^{l,as}_n$ (and T^l_n) are dominated by the stationary points where

$$\frac{\mathrm{d}t_\infty}{\mathrm{d}\lambda}(\lambda_0) = 2\pi\sigma_\infty^{(l)}(\lambda_0) = 0. \tag{2.35}$$

These equations have finite complex solutions when the mass gap is non-zero (see [2] and [3] and § 4). When the model is gapless, $\lambda_0 = \infty$ (see [4] and § 4) and the integral (2.33) is dominated for large n by its endpoints. In the former case ($\lambda_0 < \infty$), T^l_n is exponentially small in n and so is I_N in N, i.e.

$$I_N \sim |\exp(\mathrm{i} t_\infty(\lambda_0))|^N \tag{2.36}$$

(plus power corrections). On the contrary, one finds power-like behaviour plus possible logarithmic corrections in the gapless regime.

The method exposed here holds for any NBA state formed with real roots and eventually a finite number of complex roots.

3. Integrable theories and semisimple Lie algebras

We give in this section a non-exhaustive description of integrable models where the methods of § 2 apply.

A deep connection exists between integrable theories and simple Lie algebras [1, 12]. It is possible to associate an integrable vertex model to each representation of a simple Lie algebra. These models are invariant under the corresponding Lie group G, since this R matrix obeys

$$[R(\theta), g \otimes g] = 0. \tag{3.1}$$

Moreover, the structure of their BAE looks like the one of their respective Dynkin diagram. It must be noticed that the BAE are known to lead to the eigenvectors and eigenvalues of the transfer matrix only for a subset of models: those associated to U(N) [1c, 15, 19], Sp($2N$) [20] and SO($2N$) [21] and some others. However, it is plausible to assume the validity of these statements for all the semisimple Lie algebras. Moreover, the Dynkin diagram structure of the BAE is not restricted to non-Abelian invariant models fulfilling (3.1). In the case of the model of [1e] and [15], $G = U(1)^{q-1} \times Z_q$ and the NBAE have the structure of the A_{q-1} Dynkin diagram, i.e. the same structure as in the isotropic limit where $G = SU(q)$.

Let us briefly describe the BAE for the solutions of the non-Abelian symmetric models. We refer for the derivation of these equations (when they exist already) to [1, 15, 19–21].

The eigenvalues of the transfer matrix can be written as a sum of terms. The dominant one in the infinite volume limit ($N \to \infty$) is

$$\lambda_\omega(\theta, \{\lambda^{(j)}\}) = \prod_{a=1}^{N} \prod_{k=1}^{r} \prod_{j_k=1}^{p_k} \frac{\mathrm{i}(\theta - \theta_a) + \lambda_{j_k}^{(h)} - \mathrm{i}(\omega_a, \alpha_k)}{\mathrm{i}(\theta - \theta_a) + \lambda_{j_k}^{(k)} + \mathrm{i}(\omega_a, \alpha_k)} \tag{3.2}$$

for θ in the vicinity of $\theta = 0$, $|\theta| < \theta_0$. Here $\theta_1, \ldots, \theta_N$ are given numbers describing inhomogeneities in the lattice [1c, 22]. The ω_a are fundamental weights and the α_k are the simple roots of the associated Lie algebra, whose rank is r. (α, β) stands for the usual inner product in root space. The $\lambda_{j_k}^{(k)} (1 \leq j_k \leq p_j,\ 1 \leq k \leq r)$ are solutions of the NBAE. They admit the Lie algebraic expression

$$\prod_{k=1}^{r} \prod_{j_k=1}^{p_k} \frac{\lambda_{j_i}^{(i)} - \lambda_{j_k}^{(k)} - \mathrm{i}(\alpha_i, \alpha_k)}{\lambda_{j_i}^{(i)} - \lambda_{j_k}^{(k)} + \mathrm{i}(\alpha_i, \alpha_k)} = \prod_{a=1}^{N} \frac{\lambda_{j_i}^{(i)} - \theta_a - \mathrm{i}(\omega_a, \alpha_i)}{\lambda_{j_i}^{(i)} - \theta_a + \mathrm{i}(\omega_a, \alpha_i)}. \tag{3.3}$$

Here the upper indices (i) label the steps in the nested Bethe ansatz. Each step is associated to a simple root α_i. The structure of (3.3) coincides with the respective Dynkin diagram: when two roots, say α_l and α_i, are orthogonal, their associated parameters $\lambda_{j_l}^{(l)}$ and $\lambda_{j_i}^{(i)} (1 \leq j_l \leq p_l,\ 1 \leq j_i \leq p_i)$ are not directly coupled through (3.3) since $(\alpha_i, \alpha_l) = 0$. It must be noticed that due to the orthogonality of fundamental weights and simple roots [23]

$$(\omega_a, \alpha_i) = \tfrac{1}{2} \delta_{ai} (\alpha_i, \alpha_i). \tag{3.4}$$

The normalisation of the simple roots can be absorbed as a multiplicative factor on the $\lambda_{j_i}^{(i)}$.

In § 2 we dealt with the homogeneous models where $\theta_a = 0$ and $\omega_a = \omega$ for $1 \leq a \leq N$. Then

$$\phi(z, \alpha) = \mathrm{i} \log \frac{z + \mathrm{i}\alpha}{z - \mathrm{i}\alpha} \tag{3.5}$$

in the equations of § 2. We have, for $|\mathrm{Im}\, z| < \alpha$,

$$\phi(z, \alpha) = 2 \int_0^\infty \mathrm{d}k \frac{\sin kz}{k} \mathrm{e}^{-k\alpha} + \pi. \tag{3.6}$$

Here we take the cut of the logarithim in (3.5) such that $\phi(x, \alpha)$ is a continuous function for real x and $\phi(0, \alpha) = \pi$. It must be noticed that

$$\frac{\partial \phi}{\partial z}(z, \alpha)\bigg|_{\alpha=0} = 0.$$

Then (2.14*a*)-(2.16) and (3.6) yield for the kernel of the NBAE (2.14)

$$\hat{K}_{jl}(x) = -\text{sgn}[(\alpha_j, \alpha_l)] \exp[-|(\alpha_j, \alpha_l)x|] \tag{3.7}$$

with $\text{sgn}(0) \equiv 0$. Therefore, one gets for the resolvent from (2.18)

$$[\hat{R}^{-1}(x)]_{jl} = 1 + \text{sgn}[(\alpha_j, \alpha_l)] \exp[-|(\alpha_j, \alpha_l)x|]. \tag{3.8}$$

Hence, one has to invert this $r \times r$ matrix. This is not a formidable problem since it is a sparse matrix [24] whose characteristic diagram is precisely the Dynkin diagram of the corresponding Lie algebra. Explicit formulae for $R_{jl}(x)$ can be derived for each Lie algebra. One has for A_n

$$\hat{R}_{jl}(x) = e^{-|x|} \frac{\sinh x(n+1-l_>) \sinh(xl_<)}{\sinh x(n+1) \sinh x}. \tag{3.9}$$

For D_n, see [10] and [21]. For E_6, E_7 and E_8, no general formula is known but $R_{jl}(x)$ can be calculated explicitly.

For non-simply laced Lie algebras, the ground state is formed by complex roots and hence the treatment of § 2 needs to be generalised.

In the $U(1)^q (q \geq 2)$ symmetric model of [1e] and [15], one finds an A_{q-1} structure for the NBAE although the model is not $SU(q)$ invariant. One must take for $\phi(z, \alpha)$

$$\phi(z, a) = \text{i} \log \frac{\sin(z + \text{i}\alpha)}{\sin(z - \text{i}\alpha)} \tag{3.10}$$

or

$$\phi(z, \alpha) = \text{i} \log \frac{\sinh(z + \text{i}\alpha)}{\sinh(z - \text{i}\alpha)} \tag{3.11}$$

depending on the anisotropy parameter, instead of (3.5). That is, the NBAE are given by (2.2) and (2.3) with (3.2) given by (3.10) or (3.11) and the Lie algebra parameters ω_k, ω_{jk} of A_{q-1}.

Let us conclude this section by deriving a few formulae that will be needed in the finite-size calculations of the next section. We start with the asymptotic behaviour of the vacuum density of roots $\sigma_\infty^{(l)}(\lambda)$. One finds from the Fourier representations (2.18)-(2.20)

$$\sigma_\infty^{(l)}(\lambda) = \sum_{k=1}^{r} \int_{-\infty}^{+\infty} \frac{dx}{2\pi} A(x, \omega_k)[1 - \hat{K}(x)]^{-1}_{lk} e^{i\lambda x}. \tag{3.12}$$

Here $A(x, \alpha)$ is the Fourier transform of $\phi'(x, \alpha)$. So

$$A(x, \alpha) = e^{-|x\alpha|} \tag{3.13}$$

when (3.6) is used and

$$A(x, \alpha) = \frac{\sinh x(\frac{1}{2}\pi - \alpha)}{\sinh \frac{1}{2}x\pi} \tag{3.14}$$

when (3.11) holds. Equation (3.12) tells us that the large-λ behaviour of $\sigma_\infty^{(l)}(\lambda)$ is determined by the zeros of $\det[1 - \hat{K}(x)]$ closer to the real axis. These values are clearly l independent. One finds from (3.12) by the residue method

$$\sigma_\infty^{(l)}(\lambda) \underset{|\lambda| \to \infty}{=} \frac{\kappa}{\pi} m_l e^{-\kappa|\lambda|}[1 + O(e^{-|\lambda|\delta})] \tag{3.15}$$

Table 2. The parameters κ and m_l (mass spectrum) for the models associated to simply laced Lie algebras. For D_4, notice that $m_1 = m_\pm = \frac{1}{2}$ and then $\Delta = S_3$.

Lie algebra	κ	m_l	Δ	q_l
A_n	$\frac{2\pi}{n+1}$	$\sin\frac{\pi l}{n+1}, 1 \leqslant l \leqslant n$	$l \leftrightarrow n+1-l$	$\frac{n+1-l}{n+1}$
D_n	$\frac{\pi}{(n-1)}$	$\sin\frac{\pi l}{2(n-1)}, 1 \leqslant l \leqslant n-2$	$+ \leftrightarrow -$	$1, 1 \leqslant l \leqslant n-2$
		$m_\pm = \frac{1}{2}$	$n \neq 4$	$\frac{1}{2}, \pm$
E_6	$\pi/6$	$m_1 = m_5 = \sqrt{3}/2, m_2 = m_4 = \frac{1}{2}(3+\sqrt{3})$	$1 \leftrightarrow 5$	$q_1 = 2 - q_5 = \frac{4}{3}$
		$m_3 = \frac{3+\sqrt{3}}{\sqrt{2}}, m_6 = \sqrt{3}$	$2 \leftrightarrow 4$	$q_2 = 3 - q_4 = \frac{5}{3}$
E_7	$\pi/9$	(*)	None	$q_3 = 2, q_6 = 1$
E_8	$\pi/15$	(*)	None	

(*) These values can be extracted from [12].

where $\delta > 0$. The parameters κ and m_l are given in table 2 for all models associated to simply laced Lie algebras. κ is just 2π times the length square of shortest simple root in the normalisation where [23]

$$B(E_\alpha, E_{-\alpha}) = -1$$

and $B(x, y)$ is the Killing form. The coefficients m_l have the physical interpretation of particle masses in the context of the relativistic QFT with NBAE of the type (3.3). The chiral-invariant fermionic models [8] provide explicit realisations with such Lie algebraic structures. In addition, the mass spectrum has the invariance Δ of the corresponding Dynkin diagram, i.e. the automorphisms of the Lie algebra [23]. Δ is trivial or isomorphic to Z_2 except for D_4 where it is the permutation group of three elements S_3.

Although the detailed mass spectrum depends on the Lie algebra, it is true in all cases that m_l increases monotonically going from the ends of the Dynkin diagrams to the middle.

In the case of the $U(1)^{q-1}$ symmetric model [1e, 15], (3.15) holds with the m_l and κ of the A_n-symmetric theory renormalised by a factor $1/\gamma$.

4. Applications of finite-size computations

4.1. Evaluation of the central charge for gapless models

Let us compute the leading finite-size correction to the free energy for a system solvable by NBA in the gapless regime. When the ground state is filled with real solutions of the NBAE, the formalism of § 2 applies. As stated at the end of § 2, the leading finite-size corrections are governed by the large-λ behaviour of the integrand in (2.33) since we assume a zero gap. We know $\sigma_\infty^{(l)}(\lambda)$ for large λ from (3.14) and we obtain for $t^{(l)}(\lambda)$ (equation (2.6))

$$t_\infty^{(l)}(\lambda) \underset{\lambda \to \pm\infty}{=} 2\pi q_l \theta(\lambda) + (m_l/\pi)\,\mathrm{sgn}(\lambda)\, e^{-\kappa|\lambda|} \tag{4.1}$$

where

$$q_l = \int_{-\infty}^{+\infty} \mathrm{d}\lambda\, \sigma_\infty^{(l)}(\lambda) \tag{4.2}$$

is usually a rational number. This information is enough to compute the dominant large-n behaviour of T_n^l from (2.33). It is convenient to use the integration variable $t_l = t_\infty^{(l)}(\lambda_l)$ in (2.33). This yields for the free energy taking into account (2.28)

$$T_n^l(\theta) \underset{n\to\infty}{=} \sin(\kappa\theta)/\pi n^2 + \mathrm{O}(1/n^3) \tag{4.3}$$

where n is chosen such that $q_l n$ is an integer. Summation over α_l and l yields, from (2.28) and (2.31),

$$L_N(\theta) = -(\pi r/6N^2)\sin(\kappa\theta) \tag{4.4}$$

where we have used $\zeta(2) = \pi^2/6$. Now, the free energy of a conformally invariant model with periodic boundary conditions in box of size L is [14]

$$f_L \underset{L\to\infty}{=} f_\infty - \pi c/6L^2 + \text{smaller terms} \tag{4.5}$$

where c is the central charge value. However, one cannot blindly identify (4.5) and (4.4). One must first verify the rotational invariance at least for long distances. Let us analyse the spectrum of elementary excitations for low momentum and energy. The eigenvalue of $\log \tau(\theta)$ is, for a hole at $\phi \to -\infty$ in the lth branch ($1 \leqslant l \leqslant r$) from (2.33) and (4.1),

$$f_l(\varphi) - f_l(-\infty) \underset{\varphi\to-\infty}{=} -\frac{\mathrm{i}m_l}{\pi}\exp[\kappa(\varphi + \mathrm{i}\theta)] + \mathrm{O}(\mathrm{e}^{2\kappa\varphi}). \tag{4.6}$$

Since the momentum operator is given here by

$$P = -\mathrm{i}\log\tau(0) \tag{4.7}$$

one finds for this hole

$$p(\varphi) \underset{\varphi\to-\infty}{=} -\frac{m_e}{\pi}\mathrm{e}^{\kappa\varphi} \tag{4.8}$$

(we have subtracted the constant K_l such that $p(-\infty) = 0$). In this context, the Hamiltonian can be identified with

$$H = -\mathrm{Re}\log\tau(\theta). \tag{4.9}$$

So we find for the energy

$$\varepsilon \underset{\varphi\to-\infty}{=} -\frac{m_l}{\pi}\mathrm{e}^{\kappa\varphi}\sin(\kappa\theta) + \mathrm{O}(\mathrm{e}^{2\kappa\varphi})$$

$$\varepsilon \underset{p\to 0}{=} p\sin(\kappa\theta) + \mathrm{O}(p^2). \tag{4.10}$$

This indicates that we must renormalise the energy by a factor $1/\sin(\kappa\theta)$ in order to have a relativistic dispersion law and hence rotational invariance for long distances. After this renormalisation

$$L_N(\theta) \to \tilde{L}_N(\theta) = \frac{1}{\sin\kappa\theta}L_N(\theta). \tag{4.11}$$

Now, one finds from (4.4) and (4.11)

$$c = r \qquad (4.12)$$

i.e. the value of c equals the number of steps in the NBA. Each gapless step contributes by unity to the central charge. If there were some steps with a non-zero gap, they would give exponentially small corrections and no contribution to c. So the final result is that c equals the number of gapless steps of the NBA when the vacuum is formed by real roots.

4.2. Surface tension for a massive q(2q − 1)-vertex model

Let us consider the finite-size corrections for the $q(2q-1)$-vertex model of [1e] and [15]. In this model the links can be in q different states and the statistical weights are given in figure 1. For $\gamma \geq 1$, $2\theta + \gamma$ fixed, the model has a long-range generalised ferroelectric order and the dominant configurations are formed mostly by vertices of type c_1 (for $\theta > 0$) and some c_{q-1}. There are q different predominant patterns following one from each other by shifting by one the state of all links. This generalises the six-vertex ($q = 2$) situation [1c]. The interfacial tension between domains of this type is then related by standard arguments [1c] to the ratio between the asymptotically degenerate largest eigenvalues of the transfer matrix. In this model we have q asymptotically degenerate eigenvalues [21]. Λ^s $(0 \leq s \leq q-1)$ and

$$\lim_{N\to\infty} \frac{\Lambda_s(\theta)}{|\Lambda_s(\theta)|} = \exp(2\pi i s/q). \qquad (4.13)$$

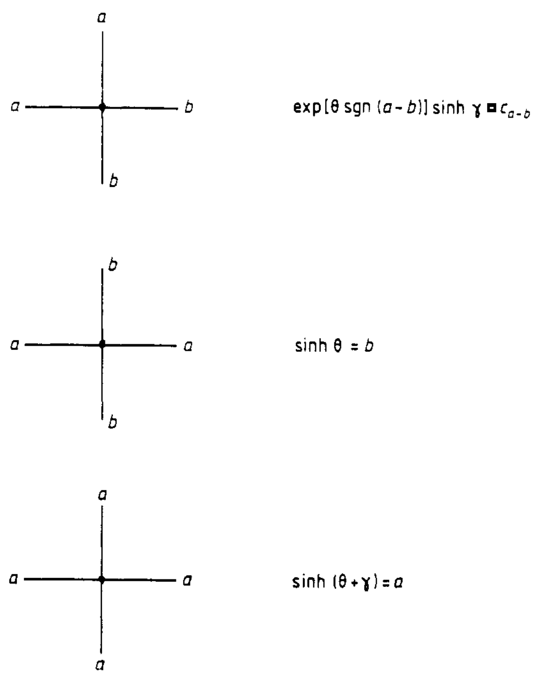


Figure 1. Statistical weights for the allowed vertex configurations in the model of [1e] and [15].

Now, one will expect

$$\Lambda_s(\theta)/\Lambda_q(\theta) = \exp(2\pi i s/q) + O(e^{-NS}) \tag{4.14}$$

where S is the interfacial tension divided by the absolute temperature.

Let us now apply the method of § 2 to explicitly compute S. We start by looking at the stationary points of the integral (2.33) and (2.34), i.e. the zeros of the density of roots [1e, 15]

$$\sigma_\infty^{(s)}(\lambda) = \frac{1}{\pi}\left(1 - \frac{s}{q} + 2\sum_{m=1}^{\infty} \frac{\cos(2m\lambda)}{\sinh(m\gamma_q)} \sinh[m\gamma(q-s)]\right).$$

This function can be expressed in terms of elliptic functions. One finds

$$\sigma_\infty^{(s)}(\lambda) = \frac{1}{2\pi}\frac{dt_\infty^{(s)}}{d\lambda} \tag{4.15}$$

$$t_\infty^{(s)}(\lambda) = \frac{1}{i}\log\frac{\theta_4(\lambda/\pi + (i\gamma/2\pi)[q-s] | i\gamma_q/\pi)}{\theta_4(\lambda/\pi - (i\gamma/2\pi)[q-s] | i\gamma_q/\pi)} + (1 - s/q)(2\lambda + \pi) \tag{4.16}$$

where $\theta_4(z|\tau)$ can be found in [25]. From general properties of the θ functions, it follows that the solutions of

$$\sigma_\infty^{(s)}(\lambda) = 0 \qquad s = 1, 2, \ldots, q-1 \tag{4.17}$$

have the form

$$\lambda_s/\pi = -\tfrac{1}{2} + i\gamma q/4\pi + i\Delta_s(\gamma)/\pi \tag{4.18}$$

where $\Delta_s(\gamma) = -\Delta_{q-s}(\gamma)$. For $q = 2$ we have $\Delta_1(\gamma) = 0$ and we recover the result of [2].

Δ_s fulfils the equation

$$\frac{\theta_2'}{\theta_2}\left(i\Delta_s + \frac{i\gamma}{4\pi}(q-2s)\right) - \frac{\theta_2'}{\theta_2}\left(i\Delta_s + \frac{i\gamma}{4\pi}(q+2s)\right) = \frac{2\pi i s}{q}. \tag{4.19}$$

This equation can easily be solved for small γ and for large γ. One finds

$$\Delta_s(\gamma) \underset{\gamma\to 0^-}{=} -\frac{\gamma q c}{\pi}\exp(-\pi^2/\gamma q)[1 - 2(1 - \tfrac{1}{3}c^2)\exp(-2\pi^2/\gamma q) + 6(1 + \tfrac{1}{5}c^4)\exp(-4\pi^2/\gamma q) + O(\exp(-6\pi^2/\gamma q))] \tag{4.20}$$

where $c \equiv \cos(\pi s/q)$,

$$\Delta_s(\gamma) \underset{\gamma\to+\infty}{=} \tfrac{1}{4}(2s-q)\gamma + \tfrac{1}{2}\log(q/s - 1) + \frac{q^2}{2}\left(\frac{\exp[-2\gamma(q-s)]}{s^2} - \frac{\exp(-2\gamma_s)}{(q-s)^2}\right) + \text{higher orders}. \tag{4.21}$$

Then, using (4.16),

$$M_s(\gamma) \equiv \exp[i t_\infty^{(s)}(\lambda_s)] \underset{\gamma\to 0^+}{=} 1 - 4T\exp(-\pi^2\gamma q) + 8T^2\exp(-2\pi^2/\gamma q) + 8T(1+T^2)\exp(-3\pi^2/\gamma q) + O(\exp(-4\pi^2/\gamma q)) \tag{4.22}$$

where $T \equiv \sin \pi s/q$ and

$$M_s(\gamma) \underset{\gamma\to+\infty}{=} \exp[-\gamma s(q-s)/q]\frac{q}{s}\left(\frac{q}{s} - 1\right)^{-1+s/q} \times \left(1 - (3q-s)\frac{\exp[-2\gamma(q-s)]}{s} + \frac{s^2 + 2qs - q^2}{(q-s)^2}e^{-2}\gamma_s + \text{higher orders}\right). \tag{4.23}$$

Now the saddle-point method applied to (2.33) and (2.34) yields

$$T_n^l(\theta) \underset{n\gg 1}{=} (M_l)^n (D_l/\sqrt{n})[1+O(1/n)] \tag{4.24}$$

where D_l is n independent. Then the finite-size corrections to the free energy (L_N) will be dominated by the larger M_l, $1 \leq l \leq q$. Equations (4.22) and (4.23) indicate that $M_1 = M_{q-1} > M_l$ for $2 \leq l \leq q-2$, so we have

$$L_N(\theta) = (D_1/\sqrt{N}) M_1(\gamma)^N. \tag{4.25}$$

Comparison of (4.14) and (4.25) yields for the interfacial tension

$$S = -\log M_1(\gamma). \tag{4.26}$$

The series (4.22) and (4.23) indicate that this quantity is always positive. It grows linearly with γ for large γ and vanishes as $\exp(-\pi q/\gamma)$ in the critical limit $\gamma \to 0^+$.

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